20 Solve each of the following for $z$.

a $z^3 - 4iz^2 + 4z - 16i = 0$ b $z^3 - 3iz^2 + 7z - 21i = 0$

c $z^3 + 2iz^2 + 5z + 10i = 0$ d $z^3 - 4iz^2 + 3z - 12i = 0$

21 a i If $P(z) = z^3 - (2 - 5i)z^2 + 3z + 15i - 6 = 0$, show that $P(2 - 5i) = 0$.

ii Find all the values of $z$ when $P(z) = 0$.

b i Given the equation $P(z) = z^3 + (-3 + 2i)z^2 + 4z + 8i - 12 = 0$, verify that $P(3 - 2i) = 0$.

ii Hence, find all values of $z$ if $P(z) = 0$.

c i Show that $-2 - 3i$ is a solution of the equation $z^3 + (2 + 3i)z^2 + 5z + 10 + 15i = 0$.

ii Find all solutions of the equation $z^3 + (2 + 3i)z^2 + 5z + 10 + 15i = 0$.

d i If $P(z) = z^3 + (-3 + 4i)z^2 + 25z + 100i - 75$ and $P(ai) = 0$, find the value(s) of the real constant $a$.

ii Hence, find all values of $z$ if $P(z) = 0$.

22 Solve each of the following for $z$.

a $z^4 + 5z^2 - 36 = 0$ b $z^4 + 4z^2 - 21 = 0$

c $z^4 - 3z^2 - 40 = 0$ d $z^4 + 9z^2 + 18 = 0$

23 a Given $P(z) = z^4 + az^3 + 34z^2 - 54z + 225$ and $P(3i) = 0$, find the value of the real constant $a$ and find all the roots.

b Given $P(z) = z^4 + 6z^3 + 29z^2 + bz + 100 = 0$ and $P(-3 - 4i) = 0$, find the value of the real constant $b$ and find all the roots.

24 a Given that $z = -2 + 3i$ is a root of the equation $2z^4 + 3z^3 + pz^2 - 77z - 39 = 0$, find the value of the real constant $p$ and all the roots.

b Given that $z = ai$ is a root of the equation $z^4 + 6z^3 + 41z^2 + 96z + 400 = 0$, find the value of the real constant $a$ and all the roots.

MASTER

25 Find a quartic polynomial with integer coefficients that has:

a $3i$ and $2 - 3i$ as the roots b $-2i$ and $-4 + 3i$ as the roots.

26 Find a quintic polynomial with integer coefficients that has:

a $-4i$, $1 + 2i$ and $-3$ as the roots b $5i$, $3 - 5i$ and 2 as the roots.

# 3.5 Subsets of the complex plane: circles, lines and rays

In previous sections, complex numbers have been used to represent points on the Argand plane. If we consider $z$ as a complex variable, we can sketch subsets or regions of the Argand plane.

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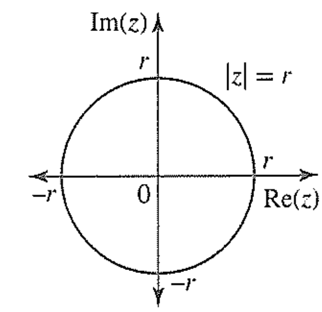
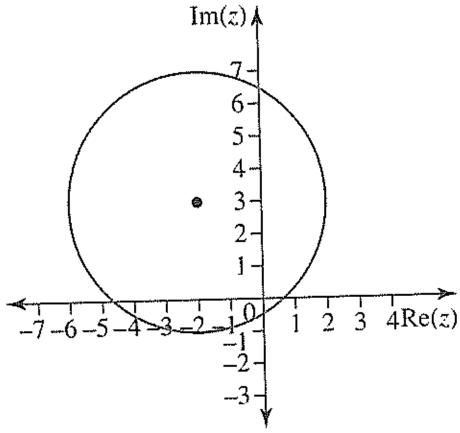
## Circles

The equation $|z| = r$ where $z = x + yi$ is given by $|z| = \sqrt{x^2 + y^2} = r$. Expanding this produces $x^2 + y^2 = r^2$. This represents a circle with centre at the origin and radius $r$. Geometrically, $|z| = r$ represents the set of points, or what is called the locus of points, in the Argand plane that are at $r$ units from the origin.



**WORKED EXAMPLE 25** Determine the Cartesion equation and sketch the graph of $\{z : |z + 2 - 3i| = 4\}$.

| THINK | WRITE/DRAW |
|---|---|
| 1 Consider the equation. | $|z + 2 - 3i| = 4$<br>Substitute $z = x + yi$:<br>$|x + yi + 2 - 3i| = 4$ |
| 2 Group the real and imaginary parts. | $|(x + 2) + i(y - 3)| = 4$ |
| 3 Use the definition of the modulus. | $\sqrt{(x + 2)^2 + (y - 3)^2} = 4$ |
| 4 Square both sides. | $(x + 2)^2 + (y - 3)^2 = 16$<br>The equation represents a circle with centre at $(-2, 3)$ and radius 4. |
| 5 Sketch and identify the graph of the Argand plane. |  |



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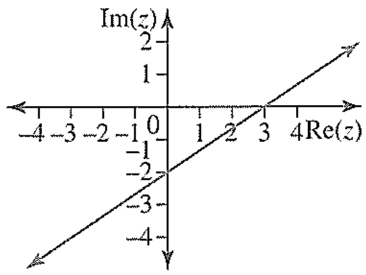
## Lines

If $z = x + yi$, then $\text{Re}(z) = x$ and $\text{Im}(z) = y$. The equation $a\text{Re}(z) + b\text{Im}(z) = c$ where $a$, $b$ and $c \in R$ represents the line $ax + by = c$.

**WORKED EXAMPLE 26** Determine the Cartesion equation and sketch the graph defined by $\{z : 2\text{Re}(z) - 3\text{Im}(z) = 6\}$.

| THINK | WRITE/DRAW |
|---|---|
| 1 Consider the equation. | $2\text{Re}(z) - 3\text{Im}(z) = 6$.<br>As $z = x + yi$, then $\text{Re}(z) = x$ and $\text{Im}(z) = y$.<br>This is a straight line with the Cartesian equation $2x - 3y = 6$. |
| 2 Find the axial intercepts. | When $y = 0$, $2x = 6 \Rightarrow x = 3$.<br>$(3, 0)$ is the intercept with the real axis.<br>When $x = 0$, $-3y = 6 \Rightarrow y = -2$.<br>$(0, -2)$ is the intercept with the imaginary axis. |

3 Identify and sketch the equation. The equation represents the line $2x - 3y = 6$.




Lines in the complex plane can also be represented as a set of points that are equidistant from two other fixed points. The equations of a line in the complex plane can thus have multiple representations.

WORKED EXAMPLE 27

**Determine the Cartesion equation and sketch the graph defined by $\{z : |z - 2i| = |z + 2|\}$.**

THINK / WRITE/DRAW

1 Consider the equation as a set of points.

$$|z - 2i| = |z + 2|$$

Substitute $z = x + yi$:

$$|x + yi - 2i| = |x + yi + 2|$$

2 Group the real and imaginary parts together.

$$|x + (y - 2)i| = |(x + 2) + yi|$$

3 Use the definition of the modulus.

$$\sqrt{x^2 + (y - 2)^2} = \sqrt{(x + 2)^2 + y^2}$$

4 Square both sides, expand, and cancel like terms.

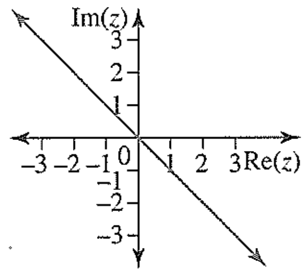
$$x^2 + y^2 - 4y + 4 = x^2 + 4x + 4 + y^2$$
$$-4y = 4x$$

5 Identify the required line.

$$y = -x$$

6 Identify the line geometrically. The line is the set of points that is equidistant from the two points (0, 2) and (−2, 0).

7 Sketch the required line.




## Intersection of lines and circles

The coordinates of the points of intersection between a straight line and a circle can be found algebraically by solving the system of equations. If there are two solutions to the equations, the line intersects the circle at two points. If there is one solution studyon

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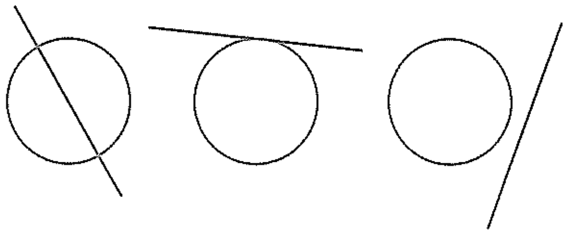
to the equation, the line and the circle touch at one point, and the line is a tangent to the circle at the point of contact. If there are no solutions to the equations, the line does not intersect the circle.



WORKED EXAMPLE 28

a **Two sets of points in the complex plane are defined by $S = \{z : |z| = 5\}$ and $T = \{z : 2\text{Re}(z) - \text{Im}(z) = 10\}$. Find the coordinates of the points of intersection between $S$ and $T$.**

b **Two sets of points in the complex plane are defined by $S = \{z : |z| = 3\}$ and $T = \{z : 2\text{Re}(z) - \text{Im}(z) = k\}$. Find the values of $k$ for which the line through $T$ is a tangent to the circle $S$.**

| THINK | WRITE |
|---|---|
| a 1 Find the Cartesian equation of $S$. | a $\lvert z\rvert = 5$<br>Substitute $z = x + yi$:<br>$\lvert x + yi\rvert = 5$ |
| 2 Use the definition of the modulus. | $\sqrt{x^2 + y^2} = 5$ |
| 3 Square both sides and identify the boundary of $S$. | (1) $x^2 + y^2 = 25$<br>$S$ is a circle with centre at the origin and radius 5. |
| 4 Find the Cartesian equation of $T$. | Substitute $z = x + yi$:<br>$\text{Re}(z) = x$ and $\text{Im}(z) = y$<br>$2\text{Re}(z) - \text{Im}(z) = 10$<br>(2) $2x - y = 10$<br>$T$ is a straight line. |
| 5 Solve equations (1) and (2) for $x$ and $y$ by substitution. | (2) $y = 2x - 10$<br>(1) $x^2 + (2x - 10)^2 = 25$ |
| 6 Expand and simplify. | $x^2 + 4x^2 - 40x + 100 = 25$<br>$5x^2 - 40x + 75 = 0$<br>$5(x^2 - 8x + 15) = 0$ |
| 7 Solve for $x$. | $x^2 - 8x + 15 = 0$<br>$(x - 5)(x - 3) = 0$<br>$x = 5$ or $x = 3$ |
| 8 Find the corresponding $y$-values. | From (2) $y = 2x - 10$,<br>when $x = 5 \Rightarrow y = 0$<br>and when $x = 3 \Rightarrow y = -4$. |
| 9 State the coordinates of the two points of intersection. | The points of intersection are $(5, 0)$ and $(3, -4)$. |
| b 1 Find the Cartesian equation of $S$. | b $\lvert z\rvert = 3$<br>Subsitute $z = x + yi$:<br>$\lvert x + yi\rvert = 3$ |

2 Use the definition of the modulus. | $\sqrt{x^2 + y^2} = 3$

3 Square both sides and identify the boundary of $S$. | (1) $x^2 + y^2 = 9$
$S$ is a circle with centre at the origin and radius 3.

4 Find the Cartesian equation of $T$. | Substitute $z = x + yi$:
$\text{Re}(z) = x$ and $\text{Im}(z) = y$
$2\text{Re}(z) - \text{Im}(z) = k$
(2) $2x - y = k$
$T$ is a straight line.

5 Solve equations (1) and (2) for $x$ and $y$ by substitution. | (2) $y = 2x - k$
(1) $x^2 + (2x - k)^2 = 9$

6 Expand and simplify. | $x^2 + 4x^2 - 4kx + k^2 = 9$
$5x^2 - 4kx + k^2 - 9 = 0$

7 If the line through $T$ is a tangent to the circle $S$, there will be only one solution for $x$. | The discriminant $\Delta = b^2 - 4ac = 0$, where $a = 5, b = -4k$ and $c = k^2 - 9$.

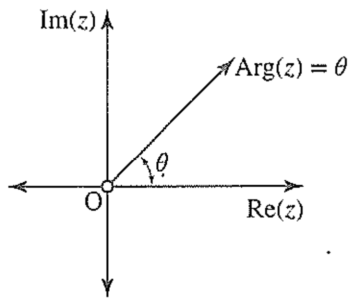
$$\begin{aligned}\Delta &= (-4k)^2 - 4 \times 5 \times (k^2 - 9)\\ &= 16k^2 - 20(k^2 - 9)\\ &= -4k^2 + 180\\ &= 4(45 - k^2)\end{aligned}$$

8 Solve the discriminant equal to zero for $k$. | $45 - k^2 = 0$
$k = \pm\sqrt{45}$

9 State the value of $k$ for which the line through $T$ is a tangent to the circle $S$. | $k = \pm 3\sqrt{5}$

## Rays

$\text{Arg}(z) = \theta$ represents the set of all points on the half-line or ray that has one end at the origin and makes an angle of $\theta$ with the positive real axis. Note that the endpoint, in this case the origin, is not included in the set. We indicate this by placing a small open circle at this point.




**WORKED EXAMPLE 29**

**Determine the Cartesion equation and sketch the graph defined by $\left\{z : \text{Arg}(z - 1 + i) = -\frac{\pi}{4}\right\}$.**

THINK | WRITE/DRAW

1 Find the Cartesian equation of the ray. | $\text{Arg}(z - 1 + i) = -\frac{\pi}{4}$
Substitute $z = x + yi$:
$\text{Arg}(x + yi - 1 + i) = -\frac{\pi}{4}$

2 Group the real and imaginary parts. | $\text{Arg}((x - 1) + (y + 1)i) = -\frac{\pi}{4}$

3 Use the definition of the argument.

$\tan^{-1}\left(\dfrac{y+1}{x-1}\right) = -\dfrac{\pi}{4}$ for $x > 1$

4 Simplify.

$$\frac{y+1}{x-1} = \tan\left(-\frac{\pi}{4}\right)$$
$$= -1 \text{ for } x > 1$$
$$y + 1 = -(x-1) \text{ for } x > 1$$

5 State the Cartesian equation of the ray.

$y = -x$ for $x > 1$.

6 Identify the point from which the ray starts.

The ray starts from the point $(1, -1)$.

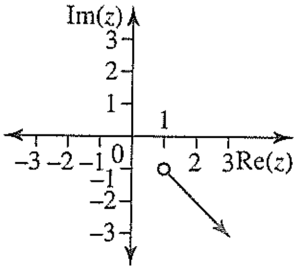
7 Determine the angle the ray makes.

The ray makes an angle of $-\frac{\pi}{4}$ with the positive real axis.

8 Describe the ray.

The point $(1, -1)$ is not included.
Alternatively, consider the ray from the origin making an angle of $-\frac{\pi}{4}$ with the positive real axis to have been translated one unit to the right parallel to the real axis, and one unit down parallel to the imaginary axis.

9 Sketch the required ray.

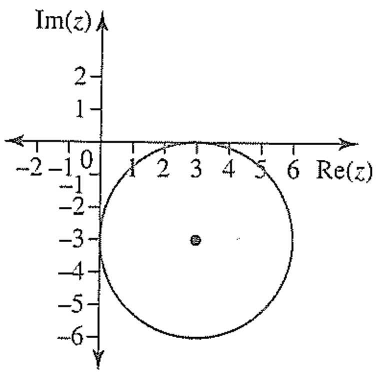



## EXERCISE 3.5 Subsets of the complex plane: circles, lines and rays

**PRACTISE**

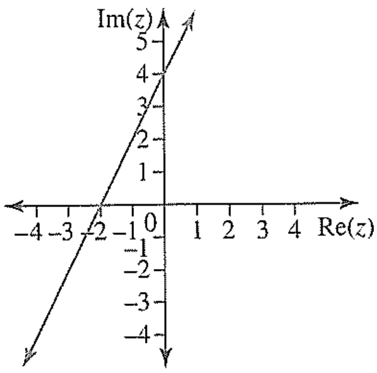
1 WE25 Sketch and describe the region of the complex plane defined by $\{z : |z - 3 + 2i| = 4\}$.

2 The region of the complex plane shown below can be described by $\{z : |z - (a + bi)| = r\}$. Find the values of $a$, $b$ and $r$.




3 WE26 Sketch and describe the region of the complex plane defined by $\{z : 4\text{Re}(z) + 3\text{Im}(z) = 12\}$.

4 The region of the complex plane shown at right can be described by $\{z : a\text{Re}(z) + b\text{Im}(z) = 8\}$. Determine the values of $a$ and $b$.




5 WE27 Sketch and describe the region of the complex plane defined by $\{z : |z + 3i| = |z - 3|\}$.

6 Sketch and describe the region of the complex plane defined by $\{z : |z - i| = |z + 3i|\}$.

7 WE28 a Two sets of points in the complex plane are defined by $S = \{z : |z| = 3\}$ and $T = \{z : 3\text{Re}(z) + 4\text{Im}(z) = 12\}$. Find the coordinates of the points of intersection between $S$ and $T$.

b Two sets of points in the complex plane are defined by $S = \{z : |z| = 4\}$ and $T = \{z : 4\text{Re}(z) - 2\text{Im}(z) = k\}$. Find the values of $k$ for which the line through $T$ is a tangent to the circle $S$.

8 a Two sets of points in the complex plane are defined by $S = \{z : |z| = \sqrt{29}\}$ and $T = \{z : 3\text{Re}(z) - \text{Im}(z) = 1\}$. Find the coordinates of the points of intersection between $S$ and $T$.

b Two sets of points in the complex plane are defined by $S = \{z : |z| = 5\}$ and $T = \{z : 2\text{Re}(z) - 3\text{Im}(z) = k\}$. Find the values of $k$ for which the line through $T$ is a tangent to the circle $S$.

9 WE29 Determine the Cartesion equation and sketch the graph defined by $\left\{z : \text{Arg}(z - 2) = \frac{\pi}{6}\right\}$.

10 Determine the Cartesion equation and sketch the graph defined by $\left\{z : \text{Arg}(z + 3i) = -\frac{\pi}{2}\right\}$.

CONSOLIDATE

11 For each of the following, sketch and find the Cartesian equation of the set, and describe the region.

a $\{z : |z| = 3\}$
b $\{z : |z| = 2\}$
c $\{z : |z + 2 - 3i| = 2\}$
d $\{z : |z - 3 + i| = 3\}$

12 Illustrate each of the following and describe the subset of the complex plane.

a $\{z : \text{Im}(z) = 2\}$
b $\{z : \text{Re}(z) + 2\text{Im}(z) = 4\}$
c $\{z : 3\text{Re}(z) + 2\text{Im}(z) = 6\}$
d $\{z : 2\text{Re}(z) - \text{Im}(z) = 6\}$

13 Sketch and describe each of the following sets, clearly indicating which boundaries are included.

a $\{z : |z - 2| = |z - 4|\}$
b $\{z : |z + 4i| = |z - 4|\}$
c $\{z : |z + 4| = |z - 2i|\}$
d $\{z : |z + 2 - 3i| = |z - 2 + 3i|\}$

14 a Show that the complex equation $\left\{z : \text{Im}\left(\frac{z - 2i}{z - 3}\right) = 0\right\}$ represents a straight line and find its equation.

b Show that the complex equation $\left\{z : \text{Re}\left(\frac{z - 2i}{z - 3}\right) = 0\right\}$ represents a circle and find its centre and radius.

15 a Find the Cartesian equation of $\{z : |z - 3| = 2|z + 3i|\}$.
b Find the locus of the set of points in the complex plane given by $\{z : |z + 3| = 2|z + 6i|\}$.
c Let $S = \{z : |z - 6| = 2|z - 3i|\}$ and $T = \{z : |z - (a + bi)| = r\}$. Given that $S = T$, find the values of $a$, $b$ and $r$.
d Let $\{z : |z + 3| = 2|z - 3i|\}$ and $T = \{z : |z - (a + bi)| = r\}$. Given that $S = T$, state the values of $a$, $b$ and $r$.

16 Four sets of points in the complex plane are defined by
$R = \{z : (z - 3 + 4i)(\bar{z} - 3 - 4i) = 25\}$, $S = \{z : |z - 3 + 4i| = 5\}$,
$T = \{z : 3\text{Re}(z) - 4\text{Im}(z) = 25\}$ and $U = \{z : |z| = |z - 6 + 8i|\}$.
a Find the Cartesian equations of $T$ and $U$ and show that $T = U$.
b Find the Cartesian equations of $S$ and $R$ and show that $S = R$.
c Sketch $S$ and $T$ on one Argand plane and find $u : S = R$ where $u \in C$.

17 a Two sets of points in the complex plane are defined by $S = \{z : |z| = 3\}$ and $T = \{z : 3\text{Re}(z) + 4\text{Im}(z) = 15\}$. Show that the line $T$ is a tangent to the circle $S$ and find the coordinates of the point of contact.
b Two sets of points in the complex plane are defined by $S = \{z : |z| = r\}$ and $T = \{z : 3\text{Re}(z) - 4\text{Im}(z) = 10\}$. Given that the line $T$ is a tangent to the circle $S$, find the value of $r$.
c Two sets of points in the complex plane are defined by $S = \{z : |z| = 2\}$ and $T = \{z : 3\text{Im}(z) - 4\text{Re}(z) = 8\}$. Find the coordinates of the points of intersection between $S$ and $T$.
d Two sets of points in the complex plane are defined by $S = \{z : |z| = 6\}$ and $T = \{z : 3\text{Re}(z) - 4\text{Im}(z) = k\}$. Find the values of $k$ for which the line through $T$ is a tangent to the circle $S$.

18 Sketch and describe the following subsets of the complex plane.
a $\left\{z : \text{Arg}(z) = \frac{\pi}{6}\right\}$
b $\left\{z : \text{Arg}(z + i) = \frac{\pi}{4}\right\}$
c $\left\{z : \text{Arg}(z - 2) = \frac{3\pi}{4}\right\}$
d $\left\{z : \text{Arg}(z + 2 - i) = -\frac{\pi}{2}\right\}$

19 a Let $S = \{z : |z| = 2\}$ and $T = \left\{z : \text{Arg}(z) = \frac{\pi}{4}\right\}$. Sketch the sets $S$ and $T$ on the same Argand diagram and find $z : S = T$.
b Let $S = \{z : |z| = 3\}$ and $T = \left\{z : \text{Arg}(z) = -\frac{\pi}{4}\right\}$. Sketch the sets $S$ and $T$ on the same Argand diagram and find $z : S = T$.
c Sets of points in the complex plane are defined by $S = \{z : |z + 3 + i| = 5\}$ and $R = \left\{z : \text{Arg}(z + 3) = -\frac{3\pi}{4}\right\}$
i Find the Cartesian equation of $S$.
ii Find the Cartesian equation of $R$.
iii If $u \in C$, find $u$ where $S = R$.

20 a Show that the complex equation $|z - a|^2 - |z - bi|^2 = a^2 + b^2$, where $a$ and $b$ are real and $b \neq 0$, represents a line.
b Show that the complex equation $|z - a|^2 + |z - bi|^2 = a^2 + b^2$, where $a$ and $b$ are real, represents a circle, and find its centre and radius.

c Show that the complex equation $3z\bar{z} + 6z + 6\bar{z} + 2 = 0$ represents a circle, and find its centre and radius.

d Consider the complex equation $az\bar{z} + bz + b\bar{z} + c = 0$ where $a$, $b$ and $c$ are real.

i If $b^2 > ac$ and $a \neq 0$, what does the equation represent?

ii If $a = 0$ and $b \neq 0$, what does the equation represent?

e Show that the complex equation $z\bar{z} + (3 + 2i)z + (3 - 2i)\bar{z} + 4 = 0$ represents a circle, and find its centre and radius.

f Consider the complex equation $az\bar{z} + \bar{b}z + b\bar{z} + c = 0$ where $a$ and $c$ are real and $b = \alpha + \beta i$ is complex. Show that the equation represents a circle provided $b\bar{b} > ac$ and $a \neq 0$, and determine the circle's centre and radius.

MASTER

21 a Show that the complex equation $\left\{z : \text{Im}\left(\frac{z - ai}{z - b}\right) = 0\right\}$ where $a$ and $b$ are real represents a straight line if $ab \neq 0$.

b Show that the complex equation $\left\{z : \text{Re}\left(\frac{z - ai}{z - b}\right) = 0\right\}$ where $a$ and $b$ are real represents a circle if $ab \neq 0$. State the circle's centre and radius.

22 Given that $c = a + bi$ where $a$ and $b$ are real:

a show that the complex equation $(z - c)(\bar{z} - \bar{c}) = r^2$ represents a circle, and find its centre and radius

b show that the complex equation $|z - c| = 2|z - \bar{c}|$ represents a circle, and find its centre and radius.

# 3.6 Roots of complex numbers

## Square roots of complex numbers

If $z^2 = x + yi = r\,\text{cis}(\theta)$, the complex number $z$ can be found using two different methods: either a rectangular method or a polar method.

### Square roots of complex numbers using rectangular form

**WORKED EXAMPLE 30**

**If $z^2 = 2 + 2\sqrt{3}i$, find the complex number $z$ using a rectangular method.**

| THINK | WRITE |
|---|---|
| 1 Expand and replace $i^2$ with $-1$. | Let $z = a + bi$ where $a, b \in R$.<br>$z^2 = a^2 + 2abi + b^2i^2$<br>$= a^2 - b^2 + 2abi$ |
| 2 Equate the real and imaginary parts. | $z^2 = a^2 - b^2 + 2abi$<br>$= 2 + 2\sqrt{3}i$<br>From the real part: $a^2 - b^2 = 2$ (1)<br>From the imaginary part: $2ab = 2\sqrt{3}$ (2) |
| 3 Solve for $b$ and substitute into (1). | From (2): $b = \frac{\sqrt{3}}{a}$. Substitute into (1):<br>$a^2 - \frac{3}{a^2} = 2$ |
| 4 Multiply by $a^2$. | $a^4 - 3 = 2a^2$<br>$a^4 - 2a^2 - 3 = 0$ |

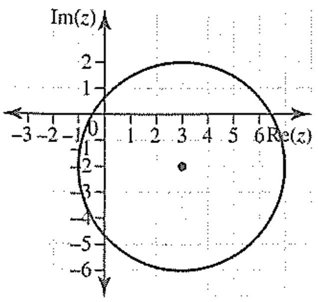
so $z^2+8z+25$ is one factor
quartic is $(z^2+4)\ (z^2+8z+25)=0$,
expanding, using CAS
$=z^4+8z^3+29z^2+32z+100$
$=0$

**26 a** $-4i,\quad 1+2i,\quad -3$
since $P(-4i)=0 \Rightarrow z^2+16$ is one factor
$\alpha=1+2i,\quad \beta=1-2i$
$\alpha+\beta=2$
$\alpha\beta=1-4i^2=5$
so $z^2-2z+5$ is a factor
$z=-3$ so $(z+3)$ is a factor
quartic is $(z+3)\ (z^2+16)(z^2-2z+5)=0$,
expanding, using CAS
$=z^5+z^4+15z^3+31z^2-16z+240$
$=0$

**b** $5i,\quad 3-5i,\quad 2$
since $P(5i)=0 \Rightarrow z^2+25$ is one factor
$\alpha=3-5i,\quad \beta=3+5i$
$\alpha+\beta=6$
$\alpha\beta=9-25i^2=34$
so $z^2-6z+34$ is a factor
$z=2$ so $(z-2)$ is a factor
quartic is $(z-2)\ (z^2+25)(z^2-6z+34)=0$,
expanding, using CAS
$=z^5-8z^4+71z^3-268z^2+1150z-1700$
$=0$

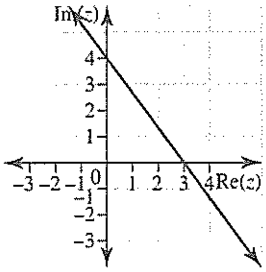
## EXERCISE 3.5 — Subsets of the complex plane: circles, lines and rays

**1** $|z-3+2i|=4$
Let $z=x+iy$
$|(x-3)+(y+2)i|=4$
$\sqrt{(x-3)^2+(y+2)^2}=4$
$(x-3)^2+(y+2)^2=16$
A circle centre $(3,-2)$ radius 4



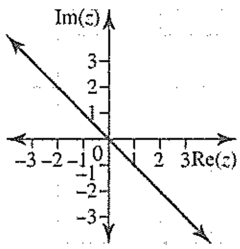

**2** Centre of the circle is at $(3,-3)$, radius is 3.
$|z-c|=r \qquad c=3-3i, r=3$
$|z-(a+bi)|=r$
$a=3, b=-3, r=3$

**3** Let $z=x+iy$
$\text{Re}(z)=x,\ \text{Im}(z)=y$
$4\,\text{Re}(z)+3\,\text{Im}(z)=12$
$4x+3y=12$
Crosses real axis $y=0\ \ x=3 \quad (3,0)$
Imag axis $x=0\ \ y=4 \quad (0,4)$

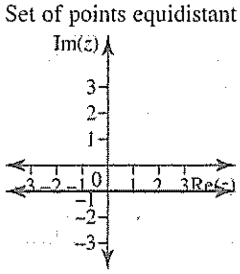



**4** line $y=mx+c$
$m=2 \quad c=4$
$y=4x+c$
$\text{Im}(z)=2\ \ \text{Re}(z)+4$
$-2\text{Re}(z)+2\text{Im}(z)=8$
$a=-4 \qquad b=2$

**5** $|z+3i|=|z-3|$
Let $z=x+iy$
$|x+(y+3)i|=|(x-3)+iy|$
$\sqrt{x^2+(y+3)^2}=\sqrt{(x-3)^2+y^2}$
Square both sides and expand
$x^2+y^2+6y+9=x^2-6x+9+y^2$
$6y=-6x$
$y=-x$
Set of points, equidistant from $(0,-3)$ to $(3,0)$




**6** $|z-i|=|z+3i|$
Let $z=x+iy$
$|x+(y-1)i|=|x+(y+3)i|$
$\sqrt{x^2+(y-1)^2}=\sqrt{x^2+(y+3)^2}$
Square both sides and expand
$x^2+y^2-2y+1=x^2+y^2+6y+9$
$6y+2y=-9+1$
$8y=-8$
$y=-1$
Set of points equidistant from $(0,1)$ to $(0,-3)$




**7 a** $S:(z:|z|=3)$
$T:(z:3\text{Re}(z)+4\,\text{Im}(z)=12)$
$S:x^2+y^2=9 \qquad (1)$
$T:3x+4y=12 \qquad (2)$
$(2)\quad 4y=12-3x$
$y=\dfrac{12-3x}{4}$ into (1)

$$x^2+\left(\frac{12-3x}{4}\right)^2=9$$
$$x^2+\frac{(12-3x)^2}{16}=9$$
$$16x^2+144-72x+9x^2=9\times16$$
$$25x^2-72x=0$$
$$x(25x-72)=0$$
$$\Rightarrow x=0 \quad \text{or} \quad x=\frac{72}{25}$$
if $x=0 \quad \Rightarrow y=3$

if $x=\frac{72}{25} \quad \Rightarrow y=\frac{1}{4}\left(12-3\times\frac{72}{25}\right)=\frac{21}{25}$

$(0,3)\left(\frac{72}{25},\frac{21}{25}\right)$

**b** $S:(z:|z|=4)$
$T:(z:4\operatorname{Re}(z)-2\operatorname{Im}(z)=k)$
$S:x^2+y^2=16 \quad (1)$
$T:4x-2y=k \quad (2)$
(2) $2y=4x-k$
$y=\frac{1}{2}(4x-k)$ into (1)
$$x^2+\left(\frac{1}{2}(4x-k)\right)^2=16$$
$$x^2+\frac{1}{4}\left(16x^2-8xk+k^2\right)=16$$
$$4x^2+16x^2-8xk+k^2=64$$
$$20x^2-8xk+\left(k^2-64\right)=0$$
For one solution $\Delta=0$
$a=20 \quad b=-8k \quad c=k^2-64$
$$b^2-4ac=0$$
$$64k^2-4\times20\left(k^2-64\right)=0$$
$$k^2-640=0$$
$$k=\pm8\sqrt{5}$$

**8 a** $S:\left(z:|z|=\sqrt{29}\right)$
$T:(z:3\operatorname{Re}(z)-\operatorname{Im}(z)=1)$
$S:x^2+y^2=29 \quad (1)$
$T:3x-y=1 \quad (2)$
$y=3x-1$ into (1)
$$x^2+(3x-1)^2=29$$
$$x^2+9x^2-6x+1=29$$
$$10x^2-6x-28=0$$
$$5x^2-3x-14=0$$
$$(5x+7)(x-2)=0$$
$$x=2 \quad \text{or} \quad -\frac{7}{5}$$
*If* $x=2\Rightarrow y=6-1=5$

$x=-\frac{7}{5} \quad y=3\times-\frac{7}{5}-1=-\frac{26}{5}$

$(2,5) \quad \left(-\frac{7}{5},-\frac{26}{5}\right)$

**b** $S:(z:|z|=5)$
$T:(z:2\operatorname{Re}(z)-3\operatorname{Im}(z)=k)$
$S:x^2+y^2=25 \quad (1)$
$T:2x-3y=k \quad (2)$
$y=\frac{1}{3}(2x-k)$ into (1)
$$x^2+\left(\frac{1}{3}(2x-k)\right)^2=25$$
$$x^2+\frac{1}{9}\left(4x^2-4xk+k^2\right)=25$$
$$9x^2+\left(4x^2-4xk+k^2\right)=225$$
$$13x^2-4xk+k^2-225=0$$
For one solution $\Delta=0$
$a=13 \quad b=-4k \quad c=k^2-225$
$$b^2-4ac=0$$
$$16k^2-4\times13\left(k^2-225\right)=0$$
$$11700-36k^2=0$$
$$-36\left(k^2-325\right)=0$$
$$k^2=325$$
$$k=\pm5\sqrt{13}$$

**9** $\left(z:\operatorname{Arg}(z-2)=\frac{\pi}{6}\right)$

Ray starts from the point (2,0) making an angle of $\frac{\pi}{6}$ with the positive real axis

$\operatorname{Arg}(z-2)=\frac{\pi}{6}$

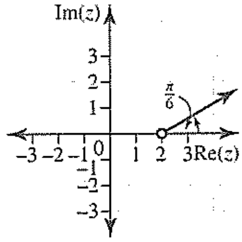
$\operatorname{Arg}(x+yi-2)=\frac{\pi}{6}$

$\operatorname{Arg}((x-2)+yi)=\frac{\pi}{6}$

$\tan^{-1}\left(\frac{y}{x-2}\right)=\frac{\pi}{6}$, for $x>2$

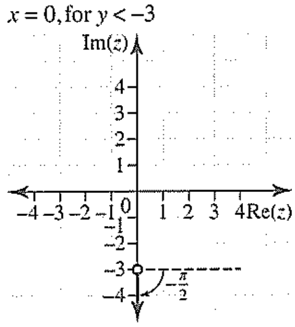
$\frac{y}{x-2}=\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$, for $x>2$

$\sqrt{3}y=(x-2)$, for $x>2$



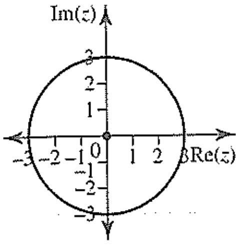

**10** $\left(z:\operatorname{Arg}(z+3i)=-\frac{\pi}{2}\right)$

Ray starts from the point (0,−3) making an angle of $-\frac{\pi}{2}$ with the positive real axis
Cartesian equation
$x=0$, for $y<-3$




**11 a** $(z:|z|=3)$, let $z=x+iy$
$$|x+iy|=3$$
$$\sqrt{x^2+y^2}=3$$
$$x^2+y^2=9$$

Circle centre origin (0,0) radius 3



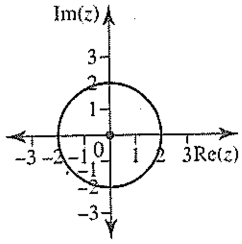

**b** $\{z : |z| = 2\}$

$|z| = 2$

$\sqrt{|x+iy|} = 2$

$\sqrt{x^2+y^2} = 2$

$x^2+y^2 = 4$




**c** $(z : |z+2-3i| = 2)$

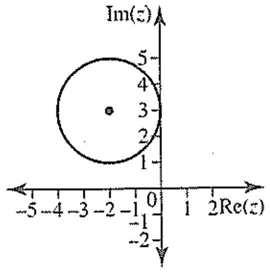
let $z = x+iy$

$|(x+2)+i(y-3)| = 2$

$\sqrt{(x+2)^2+(y-3)^2} = 4$

$(x+2)^2+(y-3)^2 = 4$

Circle centre origin(−2,3), radius 2




**d** $(z : |z-3+i| = 3)$

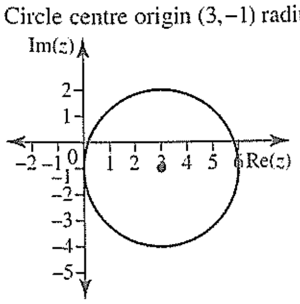
let $z = x+iy$

$|x-3+(y+1)i| = 3$

$\sqrt{(x-3)^2+(y+1)^2} = 3$

$(x-3)^2+(y+1)^2 = 9$

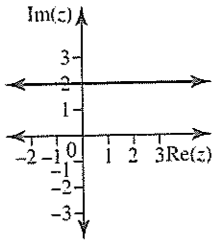
Circle centre origin (3,−1) radius 3




**12 a** $(z : \text{Im}(z) = 2)$

$z = x+iy \quad \text{Re}(z) = x, \quad \text{Im}(z) = y$

$y = 2$ line



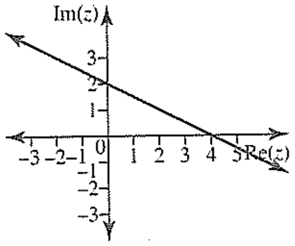

**b** $(z : \text{Re}(z) + 2\,\text{Im}(z) = 4)$

$z = x+iy \quad \text{Re}(z) = x, \quad \text{Im}(z) = y$

$x+2y = 4$

crosses imaginary axis $x = 0, \quad y = 2 \quad (0,2)$

crosses real axis $y = 0, \quad x = 4 \quad (4,0)$



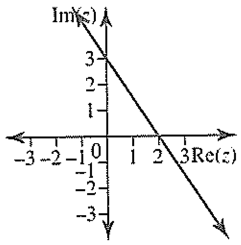

**c** $(z : 3\text{Re}(z) + 2\,\text{Im}(z) = 6)$

let $z = x+iy \quad \text{Re}(z) = x, \quad \text{Im}(z) = y$

$3x+2y = 6$ below the line

crosses imaginary axis $x = 0, \quad y = 3 \quad (0,3)$

crosses real axis $y = 0, \quad x = 2 \quad (2,0)$



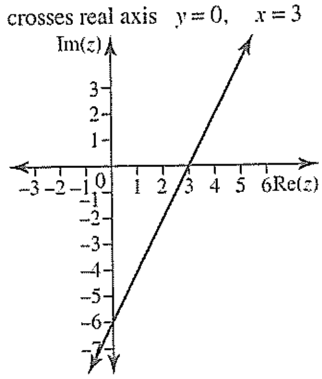

**d** $(z : 2\text{Re}(z) - \text{Im}(z) = 6)$

let $z = x+iy \quad \text{Re}(z) = x, \quad \text{Im}(z) = y$

$2x - y = 6,$

crosses imaginary axis $x = 0, \quad y = -6 \quad (0,-6)$

crosses real axis $y = 0, \quad x = 3 \quad (3,0)$




**13 a** $(z : |z-2| = |z-4|)$

let $z = x+iy$

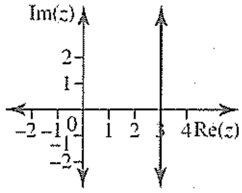
$|(x-2)+iy| = |(x-4)+iy|$

$\sqrt{(x-2)^2+y^2} = \sqrt{(x-4)^2+y^2}$

$x^2-4x+4+y^2 = x^2-8x+16+y^2$

$4x = 12$

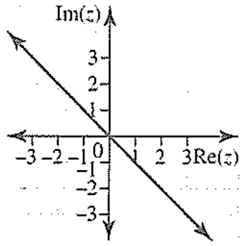
$x = 3$




**b** $\{z : |z+4i| = |z-4|\}$

let $z = x + iy$

$$|x+(y+4)i| = |(x-4)+iy|$$
$$\sqrt{x^2+(y+4)^2} = \sqrt{(x-4)^2+y^2}$$
$$x^2+y^2+8y+16 = x^2-8x+16+y^2$$
$$8x+8y=0$$
$$y=-x$$



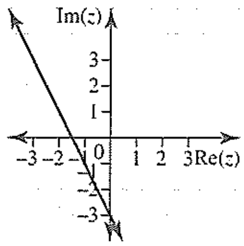

**c** $\{z : |z+4| = |z-2i|\}$

let $z = x + iy$

$$|(x+4)+iy| = |x+(y-2)i|$$
$$\sqrt{(x+4)^2+y^2} = \sqrt{x^2+(y-2)^2}$$
$$x^2+8x+16+y^2 = x^2+y^2-4y+4$$
$$8x+4y=-12$$
$$2x+y=-3$$

crosses imaginary axis $x=0 \quad y=-3 \quad (0,-3)$

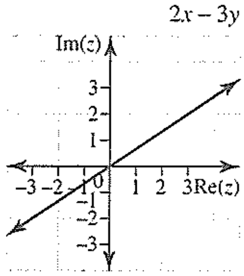
crosses real axis $y=0 \quad x=-\frac{3}{2} \quad \left(-\frac{3}{2},0\right)$




**d** $\{z : |z+2-3i| = |z-2+3i|\}$

let $z = x + iy$

$$|(x+2)+(y-3)i| = |(x-2)+(y+3)i|$$
$$\sqrt{(x+2)^2+(y-3)^2} = \sqrt{(x-2)^2+(y+3)^2}$$
$$x^2+4x+4+y^2-6y+9 = x^2-4x+4+y^2+6y+9$$
$$8x-12y=0$$
$$2x-3y=0$$




**14** Consider $\frac{z-2i}{z-3}$

Let $z = x + iy$

$$= \frac{x+(y-2)i}{(x-3)+yi} \times \frac{(x-3)-iy}{(x-3)-iy}$$
$$= \frac{x(x-3)-y(y-2)i^2+((y-2)(x-3)-xy)i}{(x-3)^2+y^2}$$
$$= \frac{x^2-3x+y^2-2y+(xy-2x-3y-xy+6)i}{(x-3)^2+y^2}$$

**a** $\text{Im}\left(\frac{z-2i}{z-3}\right)=0$

$$\Rightarrow -2x-3y+6=0$$
$$3y=-2x+6$$
$$y=-\frac{2x}{3}+2 \quad \text{line}$$

**b** $\text{Re}\left(\frac{z-2i}{z-3}\right)=0$

$$\Rightarrow x^2-3x+y^2-2y=0$$
$$\left(x^2-3x+\frac{9}{4}\right)+\left(y^2-2y+1\right)=\frac{9}{4}+1$$
$$\left(x-\frac{3}{2}\right)^2+\left(y^2-2y+1\right)=\frac{9}{4}+1$$
$$\left(x-\frac{3}{2}\right)^2+(y-1)^2=\frac{13}{4}$$

circle centre $\left(\frac{3}{2},1\right)$ radius $\frac{\sqrt{13}}{2}$

**15 a**

$$|z-3| = 2|z+3i|$$
$$|(x-3)+iy| = 2|x+(y+3)i|$$
$$\sqrt{(x-3)^2+y^2} = 2\sqrt{x^2+(y+3)^2}$$
$$(x-3)^2+y^2 = 4\left(x^2+(y+3)^2\right)$$
$$x^2-6x+9+y^2 = 4x^2+4y^2+24y+36$$
$$3x^2+6x+3y^2+24y=36$$
$$x^2+2x+y^2+8y=-9$$
$$x^2+2x+1+y^2+8y+16=-9+1+16$$
$$(x+1)^2+(y+4)^2=8$$

Circle centre $(-1,-4)$

Radius $\sqrt{8}=2\sqrt{2}$

**b**

$$|z+3| = 2|z+6i|$$
$$|(x+3)+iy| = 2|x+(y+6)i|$$
$$\sqrt{(x+3)^2+y^2} = 2\sqrt{x^2+(y+6)^2}$$
$$(x+3)^2+y^2 = 4\left(x^2+(y+6)^2\right)$$
$$x^2+6x+9+y^2 = 4x^2+4y^2+48y+144$$
$$3x^2-6x+3y^2+48y=-135$$
$$x^2-2x+y^2+16y=-45$$
$$x^2-2x+1+y^2+16y+64=-45+1+64$$
$$(x-1)^2+(y+8)^2=20$$

circle centre $(1,-8)$ radius $\sqrt{20}=2\sqrt{5}$

**c**

$$|z-6| = 2|z-3i|$$
$$|(x-6)+iy| = 2|x+(y-3)i|$$
$$\sqrt{(x-6)^2+y^2} = 2\sqrt{x^2+(y-3)^2}$$
$$(x-6)^2+y^2 = 4\left(x^2+(y-3)^2\right)$$
$$x^2-12x+36+y^2 = 4x^2+4y^2-24y+36$$
$$3x^2+12x+3y^2-24y=0$$

$$x^2+4x+y^2-8y=0$$
$$x^2+4x+4+y^2-8y+16=4+16$$
$$(x+2)^2+(y-4)^2=20$$
circle centre $(-2,4)$ radius $\sqrt{20}=2\sqrt{5}$

circle T: $|z-(a+bi)|=r$

$a=-2, b=4, r=2\sqrt{5}$

**d** $S:|z+3|=2|z-3i|$
$$|(x+3)+iy|=2|x+(y-3)i|$$
$$\sqrt{(x+3)^2+y^2}=2\sqrt{x^2+(y-3)^2}$$
$$(x+3)^2+y^2=4\left(x^2+(y-3)^2\right)$$
$$x^2+6x+9+y^2=4x^2+4y^2-24y+36$$
$$3x^2-6x+3y^2-24y=-2$$
$$x^2-2x+y^2-8y=-9$$
$$x^2-2x+1+y^2-8y+16=-9+1+16$$
$$(x-1)^2+(y-4)^2=8$$
circle centre $(1,4)$ radius $\sqrt{8}=2\sqrt{2}$

circle T: $|z-(a+bi)|=r$

$a=1, b=4, r=2\sqrt{2}$

**16 a** $T=(z:3\text{Re}(z)-4\text{Im}(z))=25$

Let $z=x+iy$

$T:3x-4y=25$

$U:(z:|z|=|z-6+8i|)$
$$|x+iy|=|(x-6)+(y+8)i|$$
$$\sqrt{x^2+y^2}=\sqrt{(x-6)^2+(y+8)^2}$$
$$x^2+y^2=x^2-12x+36+y^2+16y+64$$
$$12x-16y=100$$
$3x-4y=25$ so $T=U$

**b** $S=(z:)|z-3+4i|=5$
$$|(x-3)+(y+4)i|=5$$
$$\sqrt{(x-3)^2+(y+4)^2}=5$$
$$(x-3)^2+(y+4)^2=25$$
circle centre $(3,-4)$ radius 5

$R=(z:(z-3+4i)(\bar{z}-3-4i)=25)$

Let $c=3-4i$ $\quad \bar{c}=3+4i$ $\quad c\bar{c}=9-16i^2=25$

$z=x+iy$ $\quad \bar{z}=x-iy$ $\quad z\bar{z}=x^2+y^2$

so $(z-c)(\bar{z}-\bar{c})=25$
$$z\bar{z}-c\bar{z}-\bar{c}z+c\bar{c}=25$$
$$x^2+y^2-(3-4i)(x-iy)-(3+4i)(x+iy)=25$$
$$x^2+y^2-\left(3x-4ix-3iy+4i^2y\right)$$
$$-\left(3x+4ix+3iy+4yi^2\right)=25$$
$$25=x^2+y^2-6x+8y$$
$$25=x^2-6x+9+y^2+8y+16$$
$$25=(x-3)^2+(y+4)^2$$
circle centre $(3,-4)$ radius 5

so $S=R$

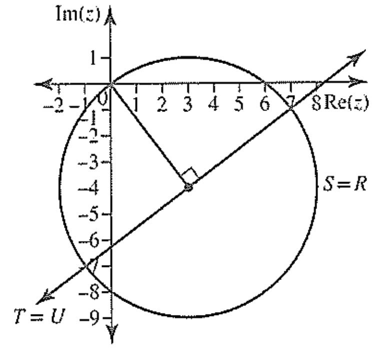
**c** $T:4y=3x-25$

$y=\dfrac{3x-25}{4}$ into $S, R$
$$(x-3)^2+\left(\frac{3x-4}{4}+4\right)^2=25$$
$$(x-3)^2+\left(\frac{3x-25+16}{4}\right)^2=25$$
$$(x-3)^2+\left(\frac{3x-9}{4}\right)^2=25$$
$$(x-3)^2+\left(\frac{3}{4}(x-3)\right)^2=25$$
$$(x-3)^2\left(1+\frac{9}{16}\right)=25$$
$$\frac{25(x-3)^2}{16}=25$$
$$(x-3)^2=16$$
$$x-3=\pm4$$
$$x=7 \text{ or } x=-1$$
when $x=7$, $y=\dfrac{21-25}{4}=-1$

$u=7-i$

when $x=-1$, $y=\dfrac{-3-25}{4}=-7$

$u=-1-7i$




**17 a** $S:|z|=3$ $\quad z=x+iy$
$$x^2+y^2=9 \quad (1)$$
$T:3\text{Re}(z)+4\text{Im}(z)=15$
$$3x+4y=15 \quad (2)$$
$$y=\frac{-3x+15}{4}$$
$$x^2+\left(\frac{-3x+15}{4}\right)^2=9$$
$$x^2+\left(\frac{3(5-x)}{4}\right)^2=9$$
$$16x^2+9\left(25-10x+x^2\right)=144$$
$$5x^2-90x-81=0$$
$$(5x-9)^2=0$$
$$x=\frac{9}{5}\Rightarrow y=\frac{15-3\times\frac{9}{5}}{4}=\frac{12}{5} \quad \left(\frac{9}{5},\frac{12}{5}\right)$$

**b** $S:|z|=r$ $\quad z=x+iy$

$x^2+y^2=r^2 \quad (1)$

$T:3\text{Re}(z)+4\text{Im}(z)=10$

$3x-4y=10 \quad (2)$
$$y=\frac{3x-10}{4} \quad (2) \text{ into } (1)$$
$$x^2+\left(\frac{3x-10}{4}\right)^2=r^2$$
$$x^2+\frac{1}{16}\left(9x^2-60x+100\right)=r^2$$
$$16x^2+9x^2-60x+100=16r^2$$
$$25x^2-60x+\left(100-16r^2\right)=0$$
For one solution $\Delta=0$

$$(60)^2 - 4\times 25\left(100-16r^2\right)=0$$
$$3600-100\left(100-16r^2\right)=0$$
$$3600-10\,000+1600r^2=0$$
$$1600\left(r^2-4\right)=0$$
$$r^2=4$$
$$r=\pm 2 \text{ but } r>0$$
$$\text{so } r=2$$

**c** $S:|z|=2$ $\quad z=x+iy$

$|x+iy|=2$

$x^2+y^2=r4 \quad (1)$

$T: 3\operatorname{Im}(z)-4\operatorname{Re}(z)=8$

$3y-4x=8 \quad (2)$

$$y=\frac{4x+8}{3} \quad (2) \text{ into } (1)$$
$$x^2+\left(\frac{4x+8}{3}\right)^2=4$$
$$x^2+\left(\frac{4}{3}(x+2)\right)^2=4$$
$$x^2+\frac{16}{9}(x+2)^2=4$$
$$9x^2+16(x+2)^2=36$$
$$9x^2+16\left(x^2+4x+4\right)=36$$
$$25x^2+64x+28=0$$
$$(5x+14)(x+2)=0$$
$$x=-2 \text{ or } -\frac{14}{25}$$

When $x=-2, y=0$ $\quad (-2,0)$

$$x=-\frac{14}{25}, y=\frac{4x-\frac{14}{25}+8}{3}=\frac{48}{25} \quad \left(-\frac{14}{25},\frac{48}{25}\right)$$

**d** $S:|z|=6$

$x^2+y^2=36 \quad (1)$

$T: 3\operatorname{Re}(z)+4\operatorname{Im}(z)=k$

$3x-4y=k \quad (2)$

$$y=\frac{3x-k}{4} \quad (2) \text{ into } (1)$$
$$x^2+\left(\frac{3x-k}{4}\right)^2=36$$
$$16x^2+9x^2-6xk+k^2=576$$
$$25x^2-6xk+\left(k^2-576\right)=0$$

For one solution $\Delta=0$

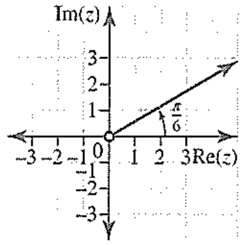
$$(6k)^2-4\times 25\left(k^2-576\right)=0$$
$$36k^2-100k^2+5760=0$$
$$-64k^2+5760=0$$
$$-64\left(k^2-90\right)=0$$
$$r=\pm 30$$

**18 a** $\left(z:\operatorname{Arg}(z)=\frac{\pi}{6}\right)$

If $z=x+iy$

$$\operatorname{Arg}(z)=\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{6}, \text{ for } x>0$$
$$\frac{y}{x}=\tan\left(\frac{\pi}{6}\right)$$
$$y=\frac{x}{\sqrt{3}} \quad \text{for } x>0$$

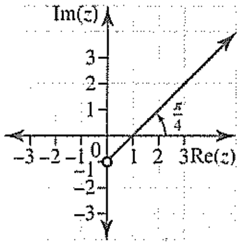
Ray from origin (not included) making an angle of 30° with the real axis




**b** $\left(z:\operatorname{Arg}(z+i)=\frac{\pi}{4}\right)$

$$\tan^{-1}\left(\frac{y+1}{x}\right)=\frac{\pi}{4}, \text{ for } x>0$$
$$\frac{y+1}{x}=\tan\left(\frac{\pi}{4}\right)$$
$$y=x-1$$

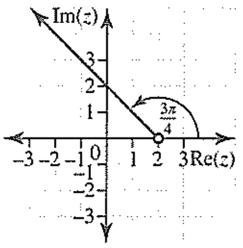
Ray from the point $(0,-1)$ (not included) making an angle of 45° with the real axis




**c** $\left(z:\operatorname{Arg}(z-2)=\frac{3\pi}{4}\right)$

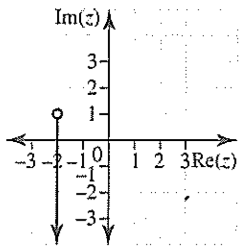
$$\operatorname{Arg}(z)=\tan^{-1}\left(\frac{y}{x-2}\right)=\frac{3\pi}{4}$$
$$\frac{y}{x-2}=\tan\left(\frac{3\pi}{4}\right)=-1$$
$$y=-(x-2)$$
$$y=2-x, \text{ for } x<2$$

Ray from the point $(2,0)$ (not included) making an angle of 135° with the real axis




**d** $\left(z:\operatorname{Arg}(z+2-i)=-\frac{\pi}{2}\right)$

Ray from the point $(-2,1)$ (not included) making an angle of $-90°$ with the real axis, it is $x=-2$, for $y<1$



**19 a** $S = \{z : |z| = 2\}$

$T = \left\{z : \text{Arg}(z) = \frac{\pi}{4}\right\}$

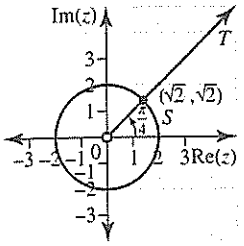
Cartesian equation of $S$ is $x^2 + y^2 = 4$

Cartesian equation of $T$ is $y = x$

$S = T$ when

$$x^2 + x^2 = 4$$
$$2x^2 = 4$$
$$x^2 = 2$$
$$x = \sqrt{2} \text{ as } x > 0$$

When $x = \sqrt{2}, y = \sqrt{2}$




**b** $S = \{z : |z| = 3\}$

$T = \left\{z : \text{Arg}(z) = -\frac{\pi}{4}\right\}$

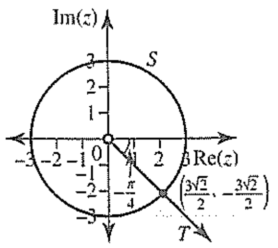
Cartesian equation of $S$ is $x^2 + y^2 = 9$

Cartesian equation of $T$ is $y = -x$

$S = T$ when

$$x^2 + x^2 = 9$$
$$2x^2 = 9$$
$$x^2 = \frac{9}{2}$$
$$x = \frac{3\sqrt{2}}{2} \text{ as } x > 0$$

When $x = \frac{3\sqrt{2}}{2}, y = -\frac{3\sqrt{2}}{2}$




**c i** $S = \{z : |z + 3 + i = 5|\}$

let $z = x + iy$

$$|(x+3) + (y+1)i| = 5$$
$$(x+3)^2 + (y+1)^2 = 25$$

circle centre $(-3, -1)$ radius 5

**ii** $R = \left\{z : \text{Arg}(z+3) = -\frac{3\pi}{4}\right\}$

Ray starting from the point $(-3, 0)$ making an angle of $-135°$ with the real axis

$$\tan^{-1}\left(\frac{y}{x+3}\right) = -\frac{3\pi}{4}$$
$$\frac{y}{x+3} = \tan\left(-\frac{3\pi}{4}\right) = 1$$
$$y = x + 3 \quad \text{for } x < -3$$

**iii**

$$S = R$$
$$y^2 + (y+1)^2 = 25$$
$$2y^2 + 2y + 1 = 25$$
$$2y^2 + 2y - 24 = 0$$
$$2(y^2 + y - 12) = 0$$
$$2(y+4)(y-3) = 0$$

If $y = -4 \Rightarrow x = -7 \quad (-7, -4) \quad u = -7 - 4i$

$y = 3 \Rightarrow x = 6$ not included at $x < -3$

**20 a** $|z - a|^2 - |z - bi|^2 = a^2 + b^2 \qquad b \neq 0, a, b \in R$

$z = x + iy$

$$|(x-a) + iy|^2 - |x + (y - bi)|^2 = a^2 + b^2$$
$$(x-a)^2 + y^2 - \left(x^2 + (y-b)^2\right) = a^2 + b^2$$
$$x^2 - 2ax + a^2 + y^2 - \left(x^2 - \left(y^2 - 2by + b^2\right)\right) = a^2 + b^2$$
$$-2ax + 2by = 2b^2$$
$$by = ax + b$$
$$y = \frac{a}{b}x + b \qquad \text{line}$$

**b** $|z - a|^2 + |z - bi|^2 = a^2 + b^2 \qquad a, b \in R$

$z = x + iy$

$$|(x-a) + iy|^2 - |x + (y-b)i|^2 = a^2 + b^2$$
$$(x-a)^2 + y^2 + \left(x^2 + (y-b)^2\right) = a^2 + b^2$$
$$x^2 - 2ax + a^2 + y^2 + x^2 + y^2 - 2by + b^2 = a^2 + b^2$$
$$x^2 - ax + y^2 - by = 0$$
$$x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} = \frac{a^2 + b^2}{4}$$
$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}$$

circle centre $\left(\frac{a}{2}, \frac{b}{2}\right)$ radius $\frac{\sqrt{a^2 + b^2}}{2}$

**c** $3z\bar{z} + 6z + 6\bar{z} + 2 = 0$

Let $z = x + iy$

$\bar{z} = x - iy$

$z\bar{z} = x^2 + y^2$

$$3z\bar{z} + 6z + 6\bar{z} + 2 = 0$$
$$3(x^2 + y^2) + 6(x + iy) + 6(x - iy) + 2 = 0$$
$$3x^2 + 3y^2 + 6x + 6iy + 6x - 6iy + 2 = 0$$
$$3x^2 + 12x + 3y^2 = -2$$
$$3(x^2 + 4x) + 3y^2 = -2$$
$$3(x^2 + 4x + 4) + 3y^2 = -2 + 12$$
$$3(x+2)^2 + 3y^2 = 10$$
$$(x+2)^2 + y^2 = \frac{10}{3}$$

circle centre $(-2, 0)$ radius $\sqrt{\frac{10}{3}} = \frac{\sqrt{30}}{3}$

**d i** $az\bar{z} + bz + b\bar{z} + c = 0$

Let $z = x + iy$

$\bar{z} = x - iy$

$z\bar{z} = x^2 + y^2$

$$3z\bar{z}+6z+6\bar{z}+c=0$$
$$a\left(x^2+y^2\right)+b(x+iy)+b(x-iy)+c=0$$
$$ax^2+ay^2+bx+biy+bx-biy+c=0$$
$$ax^2+2bx+ay^2=-c$$
$$a\left(x^2+\frac{2b}{a}x+\frac{b^2}{a^2}\right)+ay^2=-c+\frac{b^2}{a}$$
$$a\left(x+\frac{b}{a}\right)^2+ay^2=\frac{b^2-ac}{a}$$
$$\left(x+\frac{b}{a}\right)^2+y^2=\frac{b^2-ac}{a^2}$$

circle centre $\left(-\frac{b}{a},0\right)$ radius $\frac{\sqrt{b^2-ac}}{a}$

$$b^2>ac$$
$$a\neq 0$$

**ii** If $a=0$ $\quad bz+b\bar{z}+c=0$

$b(x+iy)+b(x-iy)+c=0$

$2bx+c=0$

$x=-\frac{c}{2b}$ line $b\neq 0$

**e** $z\bar{z}+(3+2i)z+(3-2i)\bar{z}+4=0$

Let $z=x+iy$

$\bar{z}=x-iy$

$z\bar{z}=x^2+y^2$

$$x^2+y^2+(3+2i)(x+iy)+(3-2i)(x+iy)+4=0$$
$$x^2+y^2+(3+2i)(x+iy)+(3-2i)(x+iy)+4=0$$
$$x^2+y^2+3x+2ix+3iy+2i^2y+3x-2ix-3iy+2i^2y+4=0$$
$$x^2+6x+9+y^2-4y+4=-4+4+9$$
$$(x+3)^2+(y-2)^2=9$$

circle centre $(-3,2)$ radius 3

**f** $az\bar{z}+bz+b\bar{z}+c=0,\quad a,c\in R\quad b=\alpha+i\beta$

Let $z=x+iy\quad \bar{z}=x-iy$

$0=a\left(x^2+y^2\right)+(\alpha-i\beta)(x+iy)+(\alpha+i\beta)(x-iy)+c$

$0=ax^2+ay^2+\alpha x-\beta ix+\alpha iy-\beta yi^2+\alpha x+\beta ix-\alpha iy-\beta yi^2+c$

$-c=ax^2+2\alpha x+ay^2+2\beta y$

$$a\left(x^2+\frac{2\alpha}{a}x+\frac{\alpha^2}{a^2}\right)+a\left(y^2+\frac{2\beta y}{a}x+\frac{\beta^2}{a^2}\right)=-c+\frac{\alpha^2}{a}+\frac{\beta^2}{a}$$
$$a\left(x+\frac{\alpha}{a}\right)^2+a\left(y+\frac{\beta}{a}\right)^2=\frac{\alpha^2+\beta^2-ac}{a}$$
$$\left(x+\frac{\alpha}{a}\right)^2+\left(y+\frac{\beta}{a}\right)^2=\frac{\alpha^2+\beta^2-ac}{a}$$
$$\left(x+\frac{\alpha}{a}\right)^2+\left(y+\frac{\beta}{a}\right)^2=\frac{\alpha^2+\beta^2-ac}{a}$$
$$=\frac{b\bar{b}-ac}{a^2}$$

$b=\alpha+i\beta\quad \bar{b}=\alpha-i\beta\quad b\bar{b}=\alpha^2+\beta^2$

circle centre $\left(-\frac{\alpha}{a},-\frac{\beta}{a}\right)$ radius $\frac{\sqrt{b\bar{b}-ac}}{a}$

provided $b\bar{b}>ac$,

$a\neq 0$

**21** Consider $\frac{z-ai}{z-b}, \quad ab \neq 0$

Let $z = x+iy$

$= \frac{x+(y-a)i}{(x-b)+yi} \times \frac{(x-b)-iy}{(x-b)-iy}$

$= \frac{x(x-b)-y(y-a)i^2+((y-a)(x-b)-xy)i}{(x-b)^2-i^2y^2}$

$= \frac{x(x-b)-y(y-a)+((y-a)(x-b)-xy)i}{(x-b)^2+y^2}$

**a** $\text{Im}\left(\frac{z-ai}{z-b}\right)=0$

$\Rightarrow (y-a)(x-b)-xy=0$

$xy-ax-by+ab-xy=0$

$ab = ax+by$

$by = -ax+ab$

$y = -\frac{a}{b}x+a \quad ab \neq 0$ line

**b** $\text{Re}\left(\frac{z-ai}{z-b}\right)=0$

$\Rightarrow x(x-b)+y(y-a)=0$

$x^2-bx+\frac{b^2}{4}+y^2-ay+\frac{a^2}{4}=\frac{a^2+b^2}{4}$

$\left(x-\frac{b}{2}\right)^2+\left(y-\frac{a}{2}\right)^2=\frac{a^2+b^2}{4}$

circle centre $\left(\frac{b}{2},\frac{a}{2}\right)$ radius $\frac{\sqrt{a^2+b^2}}{2}$

**22** Let $z = x+iy \quad \bar{z} = x-iy \quad z,c \in C$

$c = a+bi \quad \bar{c} = a-bi \quad a,b \in R$

$z\bar{z} = x^2+y^2, \quad c\bar{c} = a^2+b^2$

**a** $r^2 = z\bar{z}-c\bar{z}-\bar{c}z+c\bar{c}$

$r^2 = x^2+y^2-(a+bi)(x-iy)-(a-bi)(x+iy)+a^2+b^2$

$r^2 = x^2+y^2-(ax+bix-aiy-bi^2)-(ax-bix-aiy-byi^2)+a^2+b^2$

$r^2 = x^2-2ax+a^2+y^2-2by+b^2$

$r^2 = (x-a)^2+(y-b)^2$

circle centre $(a,b)$ radius $r$

**b**

$$|z-c| = 2|z-\bar{c}|$$

$$|(x-a)+(y-b)i| = 2|(x-a)-(y+b)i|$$

$$\sqrt{(x-a)^2+(y-b)^2} = 2\sqrt{(x-a)^2+(y+b)^2}$$

$$(x-a)^2+(y-b)^2 = 4\left((x-a)^2+(y+b)^2\right)$$

$$3(x-a)^2+4(y+b)^2-(y-b)^2=0$$

$$3(x-a)^2+4y^2+8by+4b^2-\left(y^2-2by+b^2\right)=0$$

$$3(x-a)^2+3y^2+10by+3b^2=0$$

$$(x-a)^2+y^2+\frac{10}{3}by+b^2=0$$

$$(x-a)^2+\left(y^2+\frac{10}{3}by\right)=-b^2$$

$$(x-a)^2+\left(y^2+\frac{10}{3}by+\frac{25b^2}{9}\right)=-b^2+\frac{25b^2}{9}$$

$$(x-a)^2+\left(y^2+\frac{5b}{3}\right)^2=\frac{16b^2}{9}$$

circle centre $\left(a,\frac{-5b}{3}\right)$ radius $\frac{4b}{3}$